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Classical and quantum lifetimes on some non-compact Riemann surfaces

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Abstract

This paper deals with the classical and quantum dynamics on convex co-compact surfaces. We review the recent developments of the theory and compare the asymptotic behaviour of both classical and quantum observables. We show rigorously that the classical decay rate is larger than the quantum decay rate. This is well known in the physics literature on chaotic scattering but has never been verified mathematically.

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1. Introduction

The analysis of open systems has a long history, both from the classical and quantum points of view. The most popular model of open system whose classical dynamics are 'strongly chaotic' (i.e. hyperbolic) is the convex bodies scatterer (or pinball scatterer) which has been extensively studied in both physical, see for example [3, 6–8, 11], and more mathematical literature [12, 13, 17, 20, 28, 29]. However, the lack of symmetries and the complicated dynamical geometry of these systems make them rather difficult to study and very few rigorous mathematical results are available, especially when it comes to the quantum interpretation of zeros of the so-called semi-classical dynamical zeta functions.

Motivated by the semi-classical study of Gaspard and Rice for open planar billiards [7, 8], we will describe a simple analogue of the discs scatterer coming from hyperbolic geometry where the high degree of symmetries of the manifold allows a precise (and mathematically rigorous) description of the classical and quantum physics of these objects. The semi-classical Gutzwiller–Voros zeta function turns out to be closely related to the Selberg zeta function and there is a mathematically proven [26] one-to-one correspondence between its non-trivial zeros and the scattering poles of the Laplacian, i.e. the semi-classical approximation is exact in that setting.

Because of the (relative) simplicity of their geometry, most relevant dynamical quantities are explicit, in particular we can define precisely and compute the classical and quantum escape rates of observables on these surfaces. We are thus able to check rigorously that (quantum) resonant modes have a longer lifetime than the classical observables, which is one of the main conclusions of [7, 8].

2. Convex co-compact surfaces and geodesic flow

The hyperbolic two-dimensional space can be viewed as the upper Poincaré half-plane

$$\mathbb{H}^2 = \{x + \mathrm{i}y \in \mathbb{C} : y > 0\},$$

endowed with the Riemannian metric of constant negative curvature −1 given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

An alternative isometric version (and more convenient for our purpose) of the half-plane is the disc $\mathbb{H}^2 = \{z \in \mathbb{C} : |z| < 1\}$, endowed with the metric

$$\mathrm{d}s^2 = \frac{4\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1 - |z|^2)^2}.$$

The geodesics lines (the free trajectories) are exactly the circles and lines orthogonal to the boundary. The geodesic flow ϕ_t on $\mathbb{H}^2$ is the motion (at unit speed) along the geodesic lines. The natural *phase space* of the geodesic flow is the unit tangent bundle

$$T_1\mathbb{H}^2 = \left\{(z, v) \in \mathbb{H}^2 \times \mathbb{R}^2 : \frac{2\|v\|}{1 - |z|^2} = 1\right\},$$

where $\|v\|$ denotes the standard Euclidean norm. Isometries of the hyperbolic disc are the homographic transformations of the form $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$, where $a, b \in \mathbb{C}$ and satisfy $|a|^2 - |b|^2 = 1$.

Let $\widehat{\mathbb{C}}$ denote the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The discrete groups we are interested in are the Fuchsian Schottky groups defined as follows. Let $C_1, \ldots, C_{2p}$ denote $2p$ geodesics of $\mathbb{H}^2$ and let $\mathcal{D}_1, \ldots, \mathcal{D}_{2p}$ be $2p$ Euclidean open discs such that for all $i = 1, \ldots, 2p$, the boundary of the disc $\mathcal{D}_i$ satisfies $\partial\mathcal{D}_i \cap \mathbb{H}^2 = C_i$. We assume that for all $i \neq j$, $\mathrm{Cl}(\mathcal{D}_i) \cap \mathrm{Cl}(\mathcal{D}_j) = \emptyset$, where Cl denotes the closure. Let $h_1, \ldots, h_p$ be isometries of the hyperbolic disc such that $h_i(\mathcal{D}_i) = \widehat{\mathbb{C}}\backslash\mathrm{Cl}(\mathcal{D}_{2p-i+1})$. For all $1 \leqslant i \leqslant p$, we set in addition $h_{2p-i+1} = h_i^{-1}$.

The group Γ generated by $h_1, \ldots, h_{2p}$ is a discrete group called a Fuchsian[1] Schottky group. The quotient $M = \Gamma\backslash\mathbb{H}^2$ by such a group is called a *convex co-compact surface*. It is a non-compact hyperbolic Riemann surface with infinite hyperbolic area. A natural *fundamental domain* for the action of Γ on $\mathbb{H}^2$ is simply $\mathcal{R} = \mathbb{H}^2\backslash\big(\cup_{i=1}^{2p} \mathcal{D}_i\big)$ (see figure 1).

The geodesic flow ϕ_t acting on T_1M is simply the geodesic flow on $T_1\mathbb{H}^2$ modulo the group Γ. In the example of figure 1, the surface $M = \Gamma\backslash\mathbb{H}^2$ is a pair of pants with three funnels, and the dashed geodesics γ_1 and γ_2 in $\mathbb{H}^2$ correspond to two closed geodesics on the surface, after identification by h_1 and h_2.

A convex co-compact surface can equivalently be built by considering a compact hyperbolic surface N whose boundary ∂N is a finite set of closed geodesics $\gamma_1, \ldots, \gamma_k$, and by gluing to each closed geodesic γ_i of length $l(\gamma_i)$ a funnel $F_i = (\mathbb{R}/l(\gamma_i)\mathbb{Z})_u \times \mathbb{R}_v^+$, endowed with the warped metric $\mathrm{d}s^2 = \cosh^2(v)\,\mathrm{d}u^2 + \mathrm{d}v^2$. The surface N is called the *Nielsen region* of the convex co-compact surface $M = N \cup \big(\bigcup_{i=1}^k F_i\big)$.

Under the action of the group Γ, points in $\mathbb{H}^2$ tend to the boundary $\partial\mathbb{H}^2$ which is the unit circle in the disc model. The closure of these accumulation points is a subset of $\partial\mathbb{H}^2$ called the

[1] Here Fuchsian means that this group preserves the unit disc. In the general definition of a Schottky group, one does not require this property.

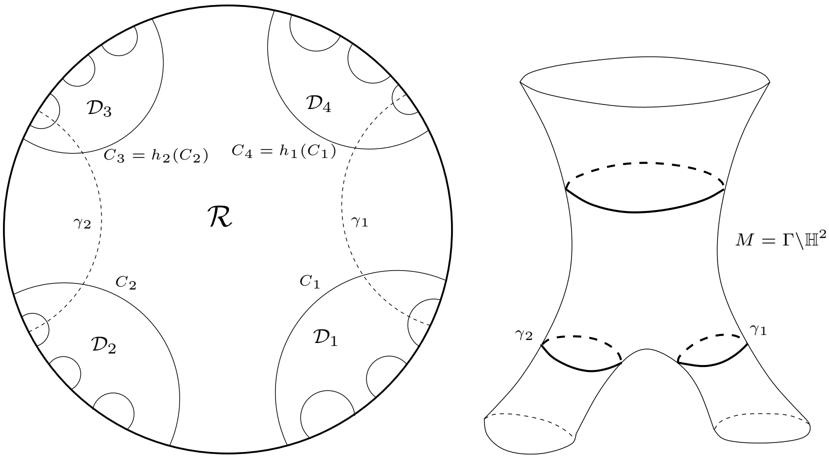

Figure 1. The fundamental domain $\mathcal{R}$ and the associated surface $M = \Gamma\backslash\mathbb{H}^2$.

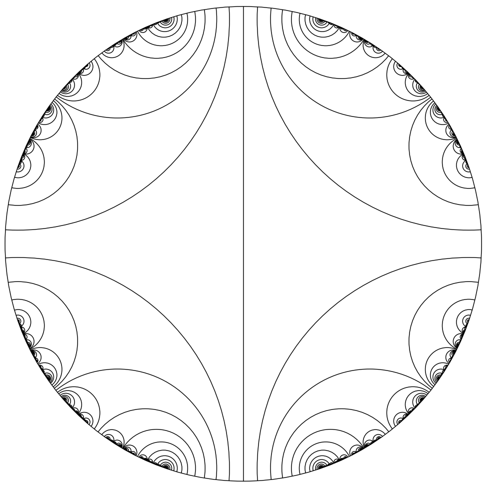

Figure 2. A typical limit set of Schottky groups.

limit set Λ of Γ. In the case of Schottky groups, when Γ is non-elementary ($p \geqslant 2$), Λ has a nice quasi self-similar structure (it is a Cantor set, see figure 2), and its Hausdorff dimension δ is non-entire, i.e. $0 < \delta < 1$.

The dimension δ can numerically be computed via the precise methods described in [19, 14].

3. Dimension of trapped set and classical escape rate

The trapped set Ω of geodesics on T_1M is defined as the compact ϕ_t-invariant set of points $x \in T_1M$ such that $\phi_t(x)$ does not escape into a funnel, i.e. remains inside the Nielsen region N for all $t \in \mathbb{R}$. There is a natural one-to-one correspondence (see [22], chapter 8) between the limit set Λ and the trapped set Ω and we have $\dim(\Omega) = 2\delta + 1$. Because of the negative curvature, the geodesic flow is hyperbolic. In addition, the set of closed geodesics (periodic orbits of the flow) is dense in Ω, i.e. axiom A of Smale is satisfied. The *topological entropy* of the flow on Ω is positive and is exactly the dimension of the limit set δ (see Sullivan [30]). This fact can alternatively be derived from the Bowen formula for the Hausdorff dimension [1], and Abramov's formula for the topological entropy of suspension flows.

Since $2\delta + 1 < 3$, the trapped set is of zero Liouville measure, thus almost all trajectories under ϕ_t escape to infinity. So, the dynamics on T_1M are of two types: the strongly chaotic motion on Ω and the Liouville-almost everywhere behaviour of points which escape to infinity. A natural question follows: what is the average escape speed? Let us be more precise. Choose a Riemannian metric g on T_1M (a natural choice is the Sasaki metric, see [23], whose associated volume Vol_g is the Liouville measure) and let d be the distance on T_1M associated with g. Given $x \in \Omega$ and $\varepsilon > 0$ we can define for all $T \geqslant 0$ the dynamical ball $B_x(\varepsilon, T)$ by

$$B_x(\varepsilon, T) = \{y \in T_1M : \forall\, 0 \leqslant t \leqslant T, d(\phi_t x, \phi_t y) \leqslant \varepsilon\}.$$

Set $\mathcal{U}_T^\varepsilon = \cup_{x\in\Omega} B_x(\varepsilon, T)$. What is the behaviour of $\mathrm{Vol}_g\big(\mathcal{U}_T^\varepsilon\big)$ as $T \to \infty$? The following result holds.

Proposition 3.1. *For all $\varepsilon > 0$ small enough, we have*

$$\limsup_{T\to+\infty} \frac{\log \mathrm{Vol}_g\big(\mathcal{U}_T^\varepsilon\big)}{T} = \delta - 1.$$

Since for all surfaces described above we have $\delta < 1$, the rate of decay is exponential and is related to the dimension of the limit set and thus to the 'size' of Ω. The limit case $\delta \to 1$ corresponds to surfaces of finite area with cuspidal ends. In that case, the trapped set is the whole unit tangent bundle and the escape rate is vanishing. We call the quantity $\alpha_c(M) = 1-\delta$ the *classical escape rate*.

Let us give a brief outline of the proof of proposition 3.1. By the volume lemma of Bowen–Ruelle [2], prop. 4.2 and 4.4, we have directly

$$\limsup_{T\to+\infty} \frac{\log \mathrm{Vol}_g\big(\mathcal{U}_T^\varepsilon\big)}{T} = P(-\lambda^u),$$

where $\lambda^u(x)$ is the local unstable Lyapounov exponent defined by

$$\lambda^u(x) = \lim_{t\to 0} \frac{1}{t} \log\left|\det\left(D\phi_t|_{E_x^u}\right)\right|,$$

E_x^u being the unstable direction of the flow. The notation $P(-\lambda^u)$ stands for the *topological pressure* of $-\lambda^u$. The topological pressure (see [23] for a review of the different formulae available for flows) is defined for continuous functions on the basic set Ω by

$$P(f) = \sup_{\mu\in\mathcal{M}_{\mathrm{inv}}} \left(h_\mu(\phi_1) + \int_\Omega f\, \mathrm{d}\mu\right),$$

where $h_\mu(\phi_t)$ is the Kolmogorov–Sinai entropy of the time one map of the flow and the supremum is taken over all ϕ_t-invariant probability measures. Because we are in a constant negative curvature case, a simple algebraic calculation involving the model $T_1M \simeq SL_2(\mathbb{R})/\Gamma$ shows that $\lambda^u \equiv 1$. We have therefore $P(-\lambda^u) = \sup_{\mu\in\mathcal{M}_{\mathrm{inv}}} h_\mu(\phi_1) - 1 = \delta - 1$, because δ is the *topological entropy* of the flow.

It is very likely that the following more meaningful result holds. Given two smooth enough and compactly supported observables f_1, f_2 on T_1M whose supports are close enough to the non-wandering set Ω, then as $t \to +\infty$,

$$\int_{T_1M} (f_1 \circ \phi_t) f_2\, \mathrm{dVol} = \mathrm{e}^{(\delta-1)t} \nu(f_2) \int_{T_1M} f_1\, \mathrm{dVol} + O(e^{-\alpha t}),$$

with $\alpha > 1 - \delta$ and ν is a measure supported by the set of stable manifolds. The fact that the left side decays exponentially for Hölder observables follows from [27]. A precise expansion as stated above should follow from an adaptation of Dolgopyat's arguments [21].

4. Scattering resonances and upper bound on their lifetime

Quantization of the free motion on M starts with the (positive) Laplace–Beltrami operator $\Delta_{\mathbb{H}^2}$ on $\mathbb{H}^2$. In the disc model, $\Delta_{\mathbb{H}^2}$ acts on smooth functions $f \in C^\infty(\mathbb{H}^2)$ by the formula

$$\Delta_{\mathbb{H}^2} f = -\frac{(1-|x+\mathrm{i}y|^2)^2}{4}\left(\partial_x^2 f + \partial_y^2 f\right).$$

Since $\Delta_{\mathbb{H}^2}$ commutes with isometries of the hyperbolic space, the Laplace–Beltrami operator Δ_M is well defined on smooth compactly supported functions $f \in C_0^\infty(\Gamma\backslash\mathbb{H}^2)$ by

$$\Delta_M(f)(\Gamma x) = \Delta_{\mathbb{H}}^2(\widetilde{f})(x),$$

where $\widetilde{f}$ is the Γ-periodic function defined on $\mathbb{H}^2$ by $\widetilde{f}(x) = f(\Gamma x)$. The operator Δ_M has a unique self-adjoint extension to a dense subdomain of the Hilbert space $L^2(M, \mathrm{d}m)$, where m is the natural Riemannian area measure on M.

The point spectrum of Δ_M is at most finite: if $\delta \leqslant \frac{1}{2}$ then it is empty and if $\delta > \frac{1}{2}$ the point spectrum is a finite subset of $\left(0, \frac{1}{4}\right)$, and the lowest eigenvalue is $\delta(1-\delta)$. The rest of the spectrum is absolutely continuous and is the half-line $\left[\frac{1}{4}, +\infty\right)$ (see the work of Patterson and Lax–Phillips [15, 16, 24]). For all $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) < 0$ (except for a finite set of values corresponding to the point spectrum) and $g \in L^2(M, \mathrm{d}m)$, the equation

$$\left(\Delta_M - \tfrac{1}{4} - \lambda^2\right) f = g,$$

has a unique solution in $L^2(M, \mathrm{d}m)$ given by

$$f(w) = \int_M G_\lambda(w, w') g(w')\, \mathrm{d}m(w'),$$

where $G_\lambda(w, w')$ is the *Green function* of $\Delta_M - \frac{1}{4} - \lambda^2$ (or equivalently, the Schwarz kernel of the resolvent). An explicit expression for $G_\lambda(w, w')$ can be obtained simply by periodizing the *free* Green function $G_\lambda^0(w, w')$ of the hyperbolic plane $\mathbb{H}^2$,

$$G_\lambda(w, w') = \sum_{\gamma\in\Gamma} G_\lambda^0(\gamma w, w'), \qquad \mathrm{Im}(\lambda) < \frac{1}{2} - \delta$$

$$G_\lambda^0(w, w') = \frac{2^{-2-2\mathrm{i}\lambda}\Gamma\left(\frac{1}{2}+\mathrm{i}\lambda\right)}{\sqrt{\pi}\,\Gamma(1+\mathrm{i}\lambda)} \cos h^{-1-2\mathrm{i}\lambda}\left[\frac{d(w,w')}{2}\right] \times F\left(\frac{1}{2}+\mathrm{i}\lambda, \frac{1}{2}+\mathrm{i}\lambda, 1+2\mathrm{i}\lambda; \cos h^2\left[\frac{d(w,w')}{2}\right]\right),$$

where $d(w, w')$ denotes the hyperbolic distance in $\mathbb{H}^2$, $\Gamma(s)$ is the usual Euler gamma function and

$$F(a, b, c; u) = 1 + \frac{a.b}{1.c}u + \frac{a(a+1)b(b+1)}{1.2c(c+1)}u^2 + \cdots$$

is the hypergeometric function.

As a function of λ, G_λ is meromorphic on $\{\mathrm{Im}(\lambda) < 0\}$, and has a meromorphic extension to $\mathbb{C}$ (see [18]), whose poles (independent of w, w') are called *resonances*. These poles have also a natural interpretation in terms of poles of a suitably defined scattering matrix (see [10]).

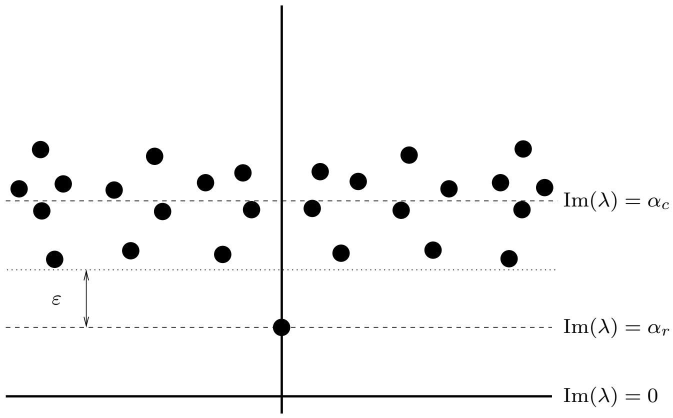

Figure 3. The non-physical upper half-plane and resonances.

In addition, G_λ has at most a finite number of poles in the lower half-plane $\{\mathrm{Im}(\lambda) < 0\}$ (the physical sheet) corresponding to the possible point spectrum of Δ_M.

The semi-classical *Gutzwiller–Voros* zeta function of these surfaces (without boundaries) is according to general theory

$$Z(\lambda) = \exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}\sum_{\gamma\in\mathcal{P}}\frac{\mathrm{e}^{-\mathrm{i}\lambda k l(\gamma)}}{\left|\det\left(P_\gamma^k - I\right)\right|^{\frac{1}{2}}}\right),$$

where $\mathcal{P}$ is the set of primitive periodic orbits γ of the geodesic flow whose period is denoted by $l(\gamma)$, and P_γ is the linear Poincaré map of γ. Since we have $\left|\det\left(P_\gamma^k - I\right)\right| = (\mathrm{e}^{kl(\gamma)} - 1)(1 - \mathrm{e}^{-kl(\gamma)})$, it follows that $Z(\lambda) = Z_M\left(\frac{1}{2} + \mathrm{i}\lambda\right)$, where $Z_M(s)$ is the Selberg zeta function of M, defined classically by the Euler product

$$Z_M(s) = \prod_{k=0}^{\infty}\prod_{\gamma\in\mathcal{P}}(1 - \mathrm{e}^{-(s+k)l(\gamma)}).$$

By the work of Patterson and Perry [26] on the divisor of $Z_M(s)$, it follows that, up to a countable set of trivial zeros on the imaginary axis located at $\mathrm{i}\left(k + \frac{1}{2}\right)$, $k \in \mathbb{N}^*$, the zeros of $Z(\lambda)$ *are the poles of the Green function* G_λ, with the same multiplicities.

In the most interesting case where there are no L^2-eigenfunctions and $\mathrm{i}\left(\delta - \frac{1}{2}\right)$ is a scattering pole, i.e. $\delta < \frac{1}{2}$, and assuming that Γ is non-elementary, we can draw a more precise picture (see figure 3):

- There are no resonances in the half-plane $\left\{\mathrm{Im}(\lambda) < \frac{1}{2} - \delta\right\}$.
- The simple scattering pole $\mathrm{i}\left(\frac{1}{2} - \delta\right)$ is isolated by a gap in the scattering spectrum, i.e. there exists $\varepsilon > 0$ such that all the other resonances lie in $\left\{\mathrm{Im}(\lambda) \geqslant \frac{1}{2} - \delta + \varepsilon\right\}$.

The leading decay rate of resonances α_r is given in the λ-plane by the smallest imaginary part of the resonances which is $\alpha_r = \frac{1}{2} - \delta < \alpha_c = 1 - \delta$. For the benefit of the reader, let us recall some of the key results proved in [21] that allow us to obtain the existence of this spectral gap in the scattering spectrum. A first observation (due to Y Sinai) is the fact that the Ruelle zeta function $\zeta(s)$ defined for $\mathrm{Re}(s) > \delta$ by

$$\zeta(s) = \prod_{\gamma\in\mathcal{P}}(1 - \mathrm{e}^{-sl(\gamma)})^{-1},$$

is related to the Selberg zeta function by

$$Z_M(s) = \prod_{k\in\mathbb{N}} \zeta(s+k)^{-1}.$$

Therefore, any knowledge on the analytic singularities of $\zeta(s)$ close to $\{\mathrm{Re}(s) = \delta\}$ implies some information on the resonances (recall that $\lambda \in \mathbb{C}$ is a resonance if and only if $\frac{1}{2} + \mathrm{i}\lambda$ is a zero of $Z_M(s)$). By the classical results of Parry and Pollicott [25], it is known that $\zeta(s)$ has a meromorphic extension to a strip $\{\delta - \overline{\varepsilon} < \mathrm{Re}(s) \leqslant \delta\}$, for some $\overline{\varepsilon} > 0$. The possible poles of $\zeta(s)$ in this strip are given by the spectrum of so-called complex transfer operators. Let us recall briefly their definition. For all $1 \leqslant i \leqslant 2p$, set $I_i = \mathrm{Cl}(\mathcal{D}_i) \cap \{|z| = 1\}$. Each I_i is a closed interval on the unit circle $\partial\mathbb{H}^2$, and if we define the *Bowen-series* map $T : I := \cup_{1\leqslant i\leqslant 2p} I_i \to \partial\mathbb{H}^2$ by

$$Tx = h_i(x), \text{ if } x \in I_i,$$

then we get a Markov expanding map (see Bowen [1]), whose dynamics encodes the action of the Schottky group Γ on $\partial\mathbb{H}^2$. Let $C^1(I)$ denotes the Banach space of C^1 functions on I endowed with its standard norm $\|.\|_{C^1}$. The transfer operators $\mathcal{L}_s$, where s is a complex parameter, are defined on $C^1(I)$ by

$$\mathcal{L}_s(f)(x) = \sum_{Ty=x} |T'(y)|^{-s} f(y).$$

A key result that follows from [25] is that $s_0 \in \{\delta - \overline{\varepsilon} < \mathrm{Re}(s) \leqslant \delta\}$ is a pole of $\zeta(s)$ if and only if 1 is an eigenvalue of $\mathcal{L}_{s_0} : C^1(I) \to C^1(I)$. In [21], we actually proved (theorem 2.3) the following estimate.

Theorem 4.1. *There exist some constants* $0 < \rho_0 < 1$, $0 < \varepsilon \leqslant \overline{\varepsilon}$, *and* $C > 0$ *such that for all* $\delta - \varepsilon < \mathrm{Re}(s) \leqslant \delta$ *and all* $|\mathrm{Im}(s)|$ *large enough, we have the contraction estimate*

$$\left\|\mathcal{L}_s^n\right\|_{C^1} \leqslant C|\mathrm{Im}(s)|^{3/2}\rho_0^n.$$

It is clear, using the spectral radius formula, that it implies that 1 cannot be an eigenvalue of $\mathcal{L}_s$ for all $\delta - \overline{\varepsilon} < \mathrm{Re}(s) \leqslant \delta$ and all $|\mathrm{Im}(s)|$ large. It is also shown in [21] that on the vertical line $\{\mathrm{Re}(s) = \delta\}$, the Ruelle zeta function $\zeta(s)$ has no poles except at $s = \delta$ which is a simple pole. Therefore at least in a narrow vertical strip close to $\{\mathrm{Re}(s) = \delta\}$, the zeta function $\zeta(s)$ has no singularities, which in view of the above discussion, shows the existence of a resonance-free strip. The proof of theorem 4.1 is based on a uniform cancellation principle discovered by Dmitry Dolgopyat in his work on Anosov flows [5]. Roughly speaking, one has to show that for large $|\mathrm{Im}(s)|$ and large n, the powers $\mathcal{L}_s^n$ (given by large sums over the preimages of T^n) exhibit enough cancellations.

Using similar techniques, we expect to be able to prove the following more physical result. Let $f \in C_0^\infty(M)$ be a smooth compactly supported initial data and let

$$u(t) = \frac{\sin(t\sqrt{\Delta_M - 1/4})}{\sqrt{\Delta_M - 1/4}} f$$

be the solution of the wave equation

$$\begin{cases} \partial_t^2 u(t) = -(\Delta_M - \frac{1}{4})u(t) \\ u(0) = 0 \\ u_t(0) = f. \end{cases}$$

Consider $\chi \in C_0^\infty(M)$, then there exists $g \in C_0^\infty(M)$ such that as $t \to +\infty$,

$$\chi\frac{\sin(t\sqrt{\Delta_M - 1/4})}{\sqrt{\Delta_M - 1/4}} f = \frac{\mathrm{e}^{(\delta-1/2)t}}{\delta - 1/2} g + O(\mathrm{e}^{-\alpha t}), \tag{1}$$

for some $\alpha > 1/2 - \delta$. An *a priori* estimate of the Green function, not yet available, is required to take advantage of the already proved information on the location of resonances. We point out that in the case of the hyperbolic cylinder (which corresponds to $p = 1$ in our notations), Christiansen and Zworski [4] have obtained a full resonance expansion of solutions of the wave equation. In that case, resonances form a lattice and the scattering matrix can be studied explicitly. We also observe that (1) is formally in agreement with the existing trace formula [9] for resonances. Indeed, let $\varphi \in C_0^\infty(\mathbb{R}_+^*)$ be a smooth, compactly supported test function, then the renormalized wave trace (the appropriate renormalization, also called 0-calculus, is designed to get rid of the divergence due to infinite volume and defines a so-called 0-volume)

$$u(t) = 0 - \mathrm{Tr}\cos\left(t\sqrt{\Delta_M - \frac{1}{4}}\right),$$

is a tempered distribution[2] which satisfies

$$u(t) = \mathrm{e}^{(\delta - \frac{1}{2})t} + \widetilde{u}(t),$$

where the remainder $\widetilde{u}(t)$ has the property

$$\int \widetilde{u}(t)\varphi(u - T)\,\mathrm{d}u = O(\mathrm{e}^{(\delta - \frac{1}{2} - \varepsilon)T}),$$

for some $\varepsilon > 0$.

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